

Characterization of Planar, 4-Connected, $K_{2,5}$ -minor-free Graphs

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Abstract

We show that every planar, 4-connected, $K_{2,5}$ -minor-free graph is the square of a cycle of even length at least six.

1 Introduction

All graphs in this paper are finite and simple. A graph is a *minor* of another graph if the first can be obtained from a subgraph of the second by contracting edges and deleting any resulting loops and parallel edges. We say that a graph G is *H -minor-free* if H is not a minor of G .

The most well-known result concerning characterizations of minor-free graphs is Wagner's demonstration in [5] that K_5 and $K_{3,3}$ -minor-free graphs are precisely the planar graphs. A related result that follows from a different formulation of Wagner's theorem is that a 2-connected graph is $K_{2,3}$ -minor-free if and only if it is K_4 or outerplanar. Other important results in this area include Dirac's [2] characterization of all K_4 -minor-free graphs and more recently, Ding and Liu's [1] description of H -minor-free graphs for all 3-connected graphs H on at most eleven edges. In [3], Ellingham et. al. provide a complete characterization of all $K_{2,4}$ -minor-free graphs. These types of questions have seen more attention since the conclusion of Robertson and Seymour's Graph Minors Project, which proved that all minor-closed families of graphs can be characterized by a finite set of forbidden minors.

In this paper we focus on $K_{2,5}$ -minor-free graphs. We suspect that this family is large and rather complex so we restrict our attention here to 4-connected planar $K_{2,5}$ -minor-free graphs. We choose 4-connected because the characterization of 5-connected $K_{2,5}$ -minor-free graphs is easy: every 5-connected graph

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either has 5 disjoint paths between a non-adjacent pair of vertices (by Menger's theorem) and hence a $K_{2,5}$ minor, or is K_6 .

To state our main result we need the following definition. The *square* of a graph G , denoted by G^2 , is a graph on the same vertex set as G , with pairs of vertices adjacent in G^2 if they are at distance at most two in G (see Fig. 3).

Theorem 1. *A graph is planar, 4-connected and $K_{2,5}$ -minor-free if and only if it is the square of a cycle of even length at least 6.*

We introduce an equivalent definition of minor, which we will use in this paper. H is a *minor* of G if for every vertex $v \in H$, there exists a connected subset of vertices $B_v \subseteq G$ called the *branch set* of v such that the branch sets of distinct vertices are disjoint and for each edge vw of H , there is an edge in G connecting B_v and B_w . Note that if G has $K_{2,t}$ or $K_{3,t}$ as a minor, we may assume that the branch sets of vertices in the part of size t each consist of only a single vertex. If G has $K_{2,5}$ with bipartition $(\{v_1, v_2\}, \{w_1, \dots, w_5\})$ as a minor, where $|B_{w_i}| = 1$, we will say that the minor is given by $\{B_{v_1}, B_{v_2}; S\}$, where $S = \bigcup_{i=1}^5 B_{w_i}$. We will analogously describe $K_{3,t}$ minors.

For a given graph G and any vertex $v \in G$, the *open neighborhood* $N(v)$ denotes the set of vertices of G adjacent to v . Similarly, for vertices $v_1, \dots, v_n \in G$, $N(v_1, \dots, v_n) = (\bigcup_{i=1}^n N(v_i)) \setminus \{v_1, \dots, v_n\}$. The *closed neighborhood* is defined to be $N[v] := N(v) \cup \{v\}$ and $N[v_1, \dots, v_n] := N(v_1, \dots, v_n) \cup \{v_1, \dots, v_n\}$.

Given a graph G , its *line graph* $L(G)$ is a graph with vertex set $V(L(G)) = E(G)$ and edge set consisting of pairs of vertices of $L(G)$ whose corresponding edges in G share a common endpoint.

2 Proof of Theorem 1

The proof of Theorem 1 uses the following result of Martinov from [4]. His result requires the following definition. A *cyclically 4-edge-connected* graph is a 3-edge-connected graph with no 3-edge-cuts that leave a cycle in each component.

Theorem 2 (Martinov, [4]). *A 4-connected graph that is 4-regular and has every edge in a triangle is either a squared cycle of length at least five or the line graph of a cubic, cyclically 4-edge-connected graph.*

Additionally, our proof uses the following lemmas, each of which describes the structure of planar, 4-connected, $K_{2,5}$ -minor-free graphs. These lemmas together with Martinov's theorem imply that 4-connected, $K_{2,5}$ -minor-free graphs must be the squares of cycles of length at least five. We then show that only even squared cycles of length at least six are both planar and 4-connected, thus finishing the proof.

Lemma 1. *For any vertex v in a 4-connected planar graph G , $N[v]$ is not a cut set.*

Proof. Suppose, to the contrary, that for some $v \in V(G)$, $N[v]$ is a cut set. Let $S \subseteq N(v)$ be a minimal cut set of the 3-connected graph $G \setminus v$. Note

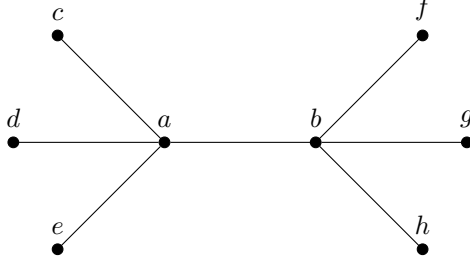


Figure 1: The neighborhood around an edge ab in G , not part of a triangle. Neighbors are shown as they appear in a planar embedding.

that $|S| \geq 3$. Let C_1 and C_2 to be two distinct components of $G \setminus (\{v\} \cup S)$. Then $(\{v\}, C_1, C_2; S)$ gives a $K_{3,|S|}$ minor, and in particular a $K_{3,3}$ minor, contradicting planarity. \square

Lemma 2. *If G is a planar, 4-connected, $K_{2,5}$ -minor-free graph, then G is 4-regular.*

Proof. To be 4-connected, G must have minimum degree at least four. Assume that some vertex v of G has degree $n > 4$. Fix a planar embedding of G and label v 's neighbors w_1, \dots, w_n , ordered clockwise around v .

Note that w_i cannot be adjacent to any w_j for $j \neq i \pm 1$, taking the indices modulo n . Otherwise, $\{v, w_i, w_j\}$ would be a 3-cut. Because $d(w_i) \geq 4$ for each i , we see that $N(w_i) \not\subseteq N[v]$.

By Lemma 1, $C = G \setminus N[v]$ is connected, so G has a $K_{2,n}$ minor given by $((C, \{v\}; \{w_1, \dots, w_n\}))$. This contradicts our choice of G , so G must be 4-regular. \square

Lemma 3. *If G is a planar, 4-connected, $K_{2,5}$ -minor-free graph, then every edge of G is in a triangle.*

Proof. Assume to the contrary that G has an edge ab not part of any triangle. We know from Lemma 2 that G is 4-regular, so a and b each have three neighbors, all distinct vertices. Fix a planar embedding of G and label these vertices as seen in Figure 1, ordered as they appear in this embedding.

Note that G cannot have exactly these eight vertices, because in particular c must have degree four and there are not three other vertices in $N[a, b]$ that c can be adjacent to. More specifically, $c \not\sim b$ because ab is not in a triangle, and $c \sim e, g, h$ would yield the 3-cuts $\{a, c, e\}$, $\{b, c, g\}$, and $\{b, c, h\}$ respectively, while G is assumed to be 4-connected. So $|V(G)| \geq 9$.

Any component of $G \setminus N[a, b]$ must be adjacent to exactly four vertices in $N(a, b)$ in order for G to be 4-connected without introducing a $K_{2,5}$ minor.

Case 1: $G \setminus N[a, b]$ has only one component, C .

Then C is nonadjacent to exactly two vertices $x, y \in N(a, b)$, which is to say

$N(x), N(y) \subseteq N[a, b]$. If x and y have a common neighbor $z \in N(a, b)$, then G has a $K_{2,5}$ minor given by $(C \cup \{z\}, \{a, b\}; N(a, b) \setminus z)$.

If x and y do not have a common neighbor, we must have $x \sim y$ and $N(a, b) \subseteq N[x, y]$ (the two additional neighbors from x and y , all distinct, will completely cover the four remaining vertices of $N(a, b)$). Then $(C, \{a, b\}, \{x, y\}; N(a, b) \setminus \{x, y\})$ gives a $K_{3,4}$ minor, contradicting the planarity of G .

Case 2: $G \setminus N[a, b]$ has more than one component.

Take any two of them, C_1 and C_2 . If C_1 and C_2 together are adjacent to every vertex of $N(a, b)$, then let $x \in N(a, b)$ be one of the two vertices adjacent to both. Then G has a $K_{2,5}$ minor given by $(\{a, b\}, \{x\} \cup C_1 \cup C_2; N(a, b) \setminus x)$. Otherwise, C_1 and C_2 must have at least three common neighbors, so let $S \subseteq N(a, b)$ consist of any three of them. Then G has a $K_{3,3}$ minor given by $(C_1, C_2, \{a, b\}; S)$, contradicting planarity.

Either way, we see that every edge of G must be in a triangle. \square

Note that the next lemma does not assume planarity.

Lemma 4. *The line graph of any cubic, 3-connected graph H has $K_{2,5}$ as a minor, unless $H \cong K_4$.*

Proof. Consider any cubic, 3-connected graph H not isomorphic to K_4 . Note that any such graph must have some edge, uv , not in a triangle. Let w, x and y, z be the (necessarily distinct) neighbors of u and v , respectively. Although the two neighbors of w other than v may not be distinct from x, y , and z , call them s and t , as in Figure 2.

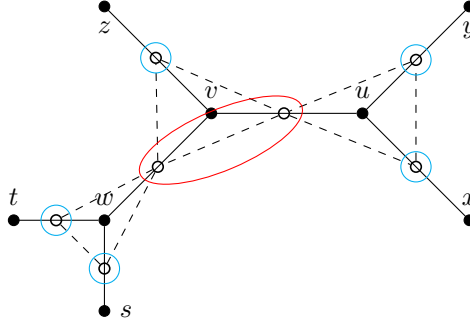


Figure 2: A neighborhood around uv in H , along with the corresponding portion of $L(H)$, highlighting the $K_{2,5}$ minor, with the branch set of the remaining vertex given by the entire rest of the line graph.

Because H is 3-connected, $H \setminus \{u, w\}$ is connected. Note that v cannot be a cut vertex of $H \setminus \{u, w\}$, so $H \setminus \{u, v, w\}$ is connected. Because in particular $x \neq y$, $H \setminus \{u, v, w\}$ must contain an edge, so it will induce a connected subgraph of $L(H)$ which avoids the edges uv, ux, uy, vz, vw, ws , and wt , which are all distinct. Then $L(H)$ has a $K_{2,5}$ minor given by $(\{uv, vw\}, L(H) \setminus$

$N[uv, vw]; N(uv, vw))$, where neighborhoods are taken in $L(H)$. See Figure 2. \square

Proof of Theorem 1. Let G be any planar, 4-connected, $K_{2,5}$ -minor-free graph. By Theorem 2, along with Lemmas 2 and 3, we see that G must be a squared cycle or the line graph of a cubic, cyclically 4-edge-connected graph. Noting that $L(K_4) \cong C_6^2$ and that a cubic, cyclically 4-edge-connected graph is, in particular, 3-connected, Lemma 4 ensures that G is a squared cycle of length at least five.

Now note that C_n^2 will have C_{n-2}^2 as a minor for all odd $n \geq 5$, so in particular will have $C_5^2 \cong K_5$ as a minor. Thus, C_n^2 is nonplanar for all such n , completing the forward direction of Theorem 1.

For the reverse direction, fix an even $n \geq 6$ and consider C_n^2 . Note that C_n^2 can be embedded in the plane with two vertex disjoint faces F_1 and F_2 of degree $\frac{n}{2}$ connected by n triangular faces. See, for example, Figure 3.

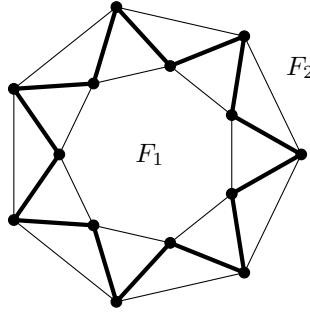


Figure 3: An embedding of C_{14}^2 , with the original 14-cycle shown in bold.

Note that C_n^2 must be 4-connected, because any cut set must contain at least two vertices from each of F_1 and F_2 .

Suppose, now, that C_n^2 has a $K_{2,5}$ minor given by $(R_1, R_2; S)$. Then without loss of generality, F_1 contains three vertices $S' \subseteq S$. Form a new graph G' from G by adding a new vertex v adjacent to these three vertices. Then G' is planar, but has a $K_{3,3}$ minor given by $(R_1, R_2, \{v\}; S')$, a contradiction. Thus, C_n^2 is planar, 4-connected, and $K_{2,5}$ -minor-free, for all even $n \geq 6$, completing the proof of Theorem 1. \square

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